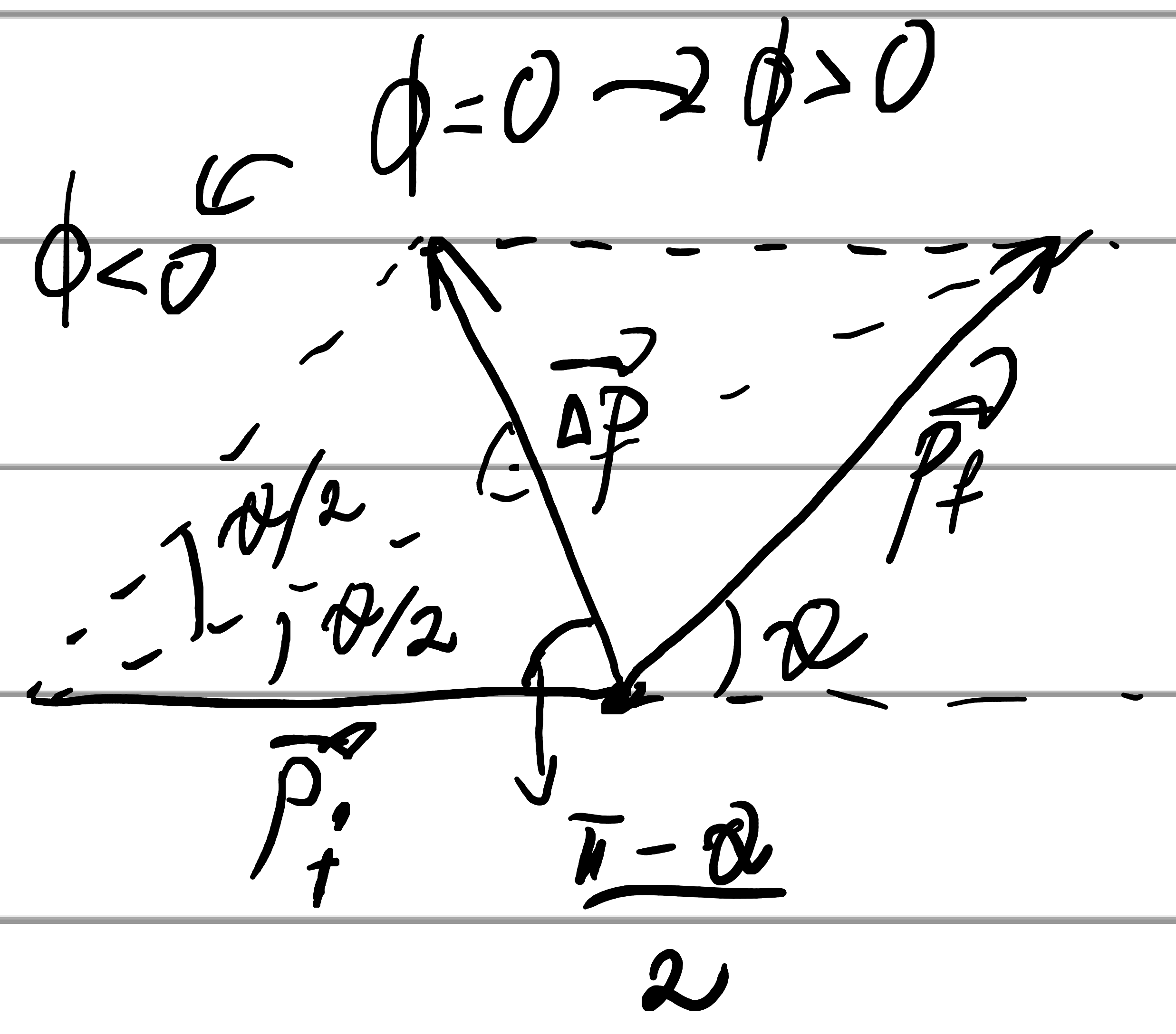
Rutherford scattering



Momentum change:

$|p_i| = |p_f| = p$

$\sin\theta/_2 = \frac{|\Delta p|/2}{p} = \frac{\Delta p}{2p}$

$\Delta p = p\cdot\sin\theta/_2$

$\vec F = \frac{d\vec p}{dt} \Rightarrow \Delta\vec p = \int \vec F dt$, $\vec F = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Z_1\cdot Z_2 e^2}{r^2}\cdot\frac{\vec r}{r}$

Trajectories are symmetric before and after the collision (wrt angle ϕ) $\Rightarrow$ $\Delta\vec p = \int \vec F dt = \int F\cos\phi\, dt$

$$\Rightarrow \boxed{\Delta p = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\int\frac{\cos\phi\, dt}{r^2}}$$

central field $\Rightarrow dr/dt = 0$

Angular momentum $\vec L = \vec r\times\vec p = mr\left(\frac{d\vec r}{dt} + r\frac{d\vec\phi}{dt}\right)$

$|\vec L| = mr^2\frac{d\phi}{dt}$ initial condition $L = mv_0 b$

$mr^2\frac{d\phi}{dt} = mv_0 b \Rightarrow \frac{dt}{r^2} = \frac{d\phi}{v_0 b}$

$$\Rightarrow \Delta p = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\int \frac{d\phi}{v_0 b}\cdot\cos\phi = \overbrace{\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\cdot\frac{1}{v_0 b}}^{A}\int_{\phi<}^{\phi>}\cos\phi\, d\phi$$

Limits of the integration: $\phi> + \phi< + \vartheta = \pi$

$\phi> = -\phi<$ $\quad \phi_< = -\frac{(\pi-\vartheta)}{2}$, $\phi_> = \frac{(\pi-\vartheta)}{2}$

$$\Delta p = A\,(\sin\phi_> - \sin\phi_<) = A.2\sin\frac{(\pi-\vartheta)}{2} = 2A\cos\vartheta/2 =$$

$$= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\cdot\frac{2}{v_0 b}\cdot\cos\vartheta/2$$

But also $\Delta p = 2p\sin\vartheta/2$

$$\Rightarrow b = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\cdot\frac{1}{v_0 p}\cot\vartheta/2 =$$ initial $p = mv_0$

$$= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\cdot\frac{1}{2T}\cdot\frac{1}{\tan\vartheta/2}$$ T - initial kinetic energy

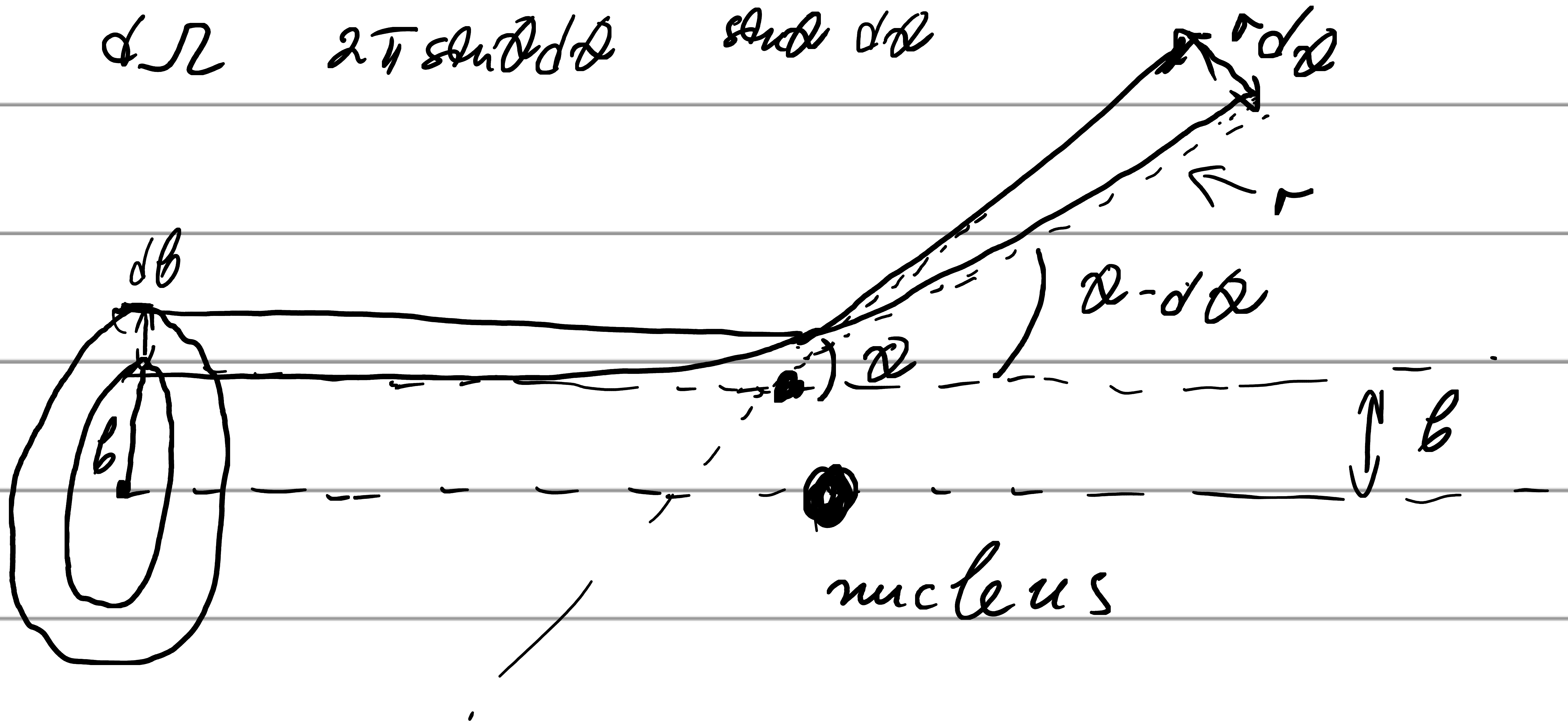
For head on collision the CDA is given by

$$T = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 \cdot d_0}$$ (kinetic energy = potential energy)

$$\Rightarrow \boxed{b = \frac{d_0}{2}\cdot\frac{1}{\tan\vartheta/2}}$$

Cross section:

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b\, db}{2\pi \sin\vartheta\, d\vartheta} = \frac{b}{\sin\vartheta}\frac{db}{d\vartheta}$$



$$\frac{d\sigma}{d\Omega} = \frac{d_0}{2 \cdot \tan\vartheta/_2 \cdot \sin\vartheta} \cdot \frac{d_0}{2} \cdot \frac{1}{\sin^2\vartheta/_2} = \left(\frac{d_0}{2}\right)^2 \cdot \frac{\cancel{\cos\vartheta/2}}{\sin^3\vartheta/_2 \cdot 2\sin\vartheta/_2 \cdot \cancel{\cos\vartheta/_2}} =$$

$$= \left(\frac{d_0}{2}\right)^2 \cdot \frac{1}{\sin^4\vartheta/_2} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 \cdot 4T} \cdot \frac{1}{\sin^4\vartheta/_2} = \left(\frac{Z_1 Z_2 \alpha}{4T}\right)^2 \cdot \frac{1}{\sin^4\vartheta/_2}$$